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(Residential Autonomous College affiliated to University of Calcutta)

M.A./M.Sc. SECOND SEMESTER EXAMINATION, MAY 2019

FIRST YEAR [BATCH 2018-20]

Date : 08/05/2019 **MATHEMATICS**

Time : 11 am – 1 pm **Paper :** VII (Linear Algebra-II) Full Marks : 50

[Answer as many questions as you possibly can. Maximum you can score in 50. Symbols have their usual meanings.]

1. Let R be an integral domain and Q be the field of fractions of R. Show that if a field F contains a subring R' isomorphic to R, then the subfield generated by R' is isomorphic to Q. (5)

2. a) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

   b) Show that the group D_8 is a split extension of $\mathbb{Z}_2$ by $\mathbb{Z}_4$. (3+3)

3. a) Give an example to show that a sub-module of a free module need not be free.

   b) What will be possible $\mathbb{Z}$-module homomorphisms of $\mathbb{Z}/2\mathbb{Z}$ into some $\mathbb{Q}$-module. (2+2)

4. Show that for any set A there is a free R-module F(A) on the set A and F(A) satisfies the following universal property : If M is any R-module and $\phi : A \to M$ is any map of sets ,then there is a unique R-module homomorphism $\Phi : F(A) \to M$ such that $\Phi(a) = \phi(a)$ for all $a \in A$. (7)

5. Show that the operations extension and contraction of ideals give a bijective correspondence:

   $$\{\text{Prime ideals P of R with } P \cap D = \phi\} \underset{c}{\overset{e}{\rightleftarrows}} \{\text{Prime ideals of } D^{-1}R\}.$$ (5)

6. Show that free modules are flat . Also show that projective modules are flat. (7)

7. $\mathbb{Z}$-modules are same as abelian groups.Justify! (2)

8. Let R be a commutative ring with 1 and let I and J be ideals of R.

   a) Prove that every element of $\frac{R}{I} \otimes_R \frac{R}{J}$ can be written as a simple tensor of the form $(1 \bmod I) \otimes (r \bmod J)$.

   b) Prove that $\frac{R}{I} \otimes_R \frac{R}{J} \cong \frac{R}{I+J}$ (2+3)

9. Let D, L and N be R-modules. Prove that $\operatorname{Hom}_R(D, L \oplus N) \cong \operatorname{Hom}_R(D, L) \oplus \operatorname{Hom}_R(D, N)$. (5)

10. Let $R = \mathbb{Z}[x]$ and let I be the ideal generated by 2 and x. Let $M = I \otimes_R I$. Show that $2 \otimes x \neq x \otimes 2$ in M. Also show that $2 \otimes 2 + x \otimes x$ is not a simple tensor. (2+2)

11. Show that $2 \otimes (1 + 2\mathbb{Z})$ is zero in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ but is non-zero in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. (2)

12. Prove that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(n, m)\mathbb{Z}$. (3)

13. Let R be a commutative ring with 1, let I be an ideal in R and let d be a multiplicatively closed subset of R containing 1 .Then prove that $\operatorname{rad}(D^{-1}I) = D^{-1}(\operatorname{rad} I)$. (3)

14. Let R be a commutative ring. Prove that $\operatorname{Hom}_R(R, M)$ and M are isomorphic as left R-modules. (2)

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